THE UNIVERSAL CHARACTER RING OF THE $(-2, 2m+1, 2n)$-PRETZEL LINK

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Abstract. We explicitly calculate the universal character ring of the $(-2, 2m+1, 2n)$-pretzel link and show that it is reduced for all integers m and n.

0. Introduction

0.1. **The character variety and the universal character ring.** The set of representations of a finitely presented group G into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits of that action. A better quotient, the algebro-geometric quotient denoted by $X(G)$ (see [LM]), has the structure of an algebraic set. There is a bijection between $X(G)$ and the set of all characters of representations of G into $SL_2(\mathbb{C})$, hence $X(G)$ is usually called the *character variety* of G. It is determined by the traces of some fixed elements $g_1, \cdots, g_k$ in G. More precisely, one can find $g_1, \cdots, g_k$ in G such that for every element g in G there exists a polynomial P_g in k variables such that for any representation $\rho : G \to SL_2(\mathbb{C})$ one has $\operatorname{tr}(\rho(g)) = P_g(x_1, \cdots, x_k)$ where $x_j := \operatorname{tr}(\rho(g_j))$. The *universal character ring* of G is then defined to be the quotient of the polynomial ring $\mathbb{C}[x_1, \cdots, x_k]$ by the ideal generated by all expressions of the form $\operatorname{tr}(\rho(u)) - \operatorname{tr}(\rho(v))$, where u and v are any two words in the letters $g_1, \cdots, g_k$ which are equal in G, c.f. [LT1]. The universal character ring of G is actually independent of the choice of $g_1, \cdots, g_k$. The quotient of the universal character ring of G by its nil-radical is equal to the ring of regular functions on the character variety $X(G)$.

0.2. **Main results.** Let $F_2 := \langle a, w \rangle$ be the free group in 2 letters a and w. The character variety of F_2 is isomorphic to $\mathbb{C}^3$ by the Fricke-Klein-Vogt theorem, see [LM]. For every word u in F_2 there is a *unique* polynomial P_u in 3 variables such that for any representation $\rho : F_2 \to SL_2(\mathbb{C})$ one has $\operatorname{tr}(\rho(u)) = P_u(x, y, z)$ where $x := \operatorname{tr}(\rho(a))$, $y := \operatorname{tr}(\rho(w))$ and $z := \operatorname{tr}(\rho(aw))$. For a word u in F_2, we denote by $\overleftarrow{u}$ the word obtained from u by writing the letters in u in reversed order. In this paper we consider the group

$$G := \langle a, w \mid r = \overleftarrow{r} \rangle,$$

where r is a word in F_2. For every representation $\rho : G \to SL_2(\mathbb{C})$, we consider x, y, and z as functions of ρ. The universal character ring of G is calculated as follows.

Theorem 1. *The universal character ring of the group $\langle a, w \mid r = \overleftarrow{r} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the principal ideal generated by the polynomial $P_{raw} - P_{\overleftarrow{r}aw}$.*

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In our joint work with T. Le on the AJ conjecture of [Ga, Ge, FGL] which relates the A-polynomial and the colored Jones polynomials of a knot, it is important to know whether the universal character ring of the knot group is reduced, i.e. whether its nilradical is zero [Le2, LT1]. So far there are a few groups for which the universal character ring is known to be reduced: free groups [Si], surface groups [CM, Si], two-bridge knot groups [PS], torus knot groups [Ma], the $(-2, 3, 2n+1)$-pretzel knot groups [LT1], and two-bridge link groups [LT2].

In the present paper we consider the $(-2, 2m+1, 2n)$-pretzel link group, where m and n are integers. As an application of Theorem 1 we will show that

Theorem 2. *(i) The fundamental group of the $(-2, 2m+1, 2n)$-pretzel link is isomorphic to the group $\langle a, w \mid r = \overleftarrow{r} \rangle$ where $r := u^{n-1}awaw^{-1}a^{-1}$ and $u := (awaw^{-1})^{1-m}w$. Hence its universal character ring is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the principal ideal generated by the polynomial*

$$P_{raw} - P_{\overleftarrow{r}aw} = (xyz + 4 - x^2 - y^2 - z^2)[(xz - y)S_{n-1}(\alpha) - (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha)],$$

where

$$\begin{aligned} \alpha &:= P_u = yS_{m-1}(\beta) - (xz - y)S_{m-2}(\beta), \\ \beta &:= P_{awaw^{-1}} = xyz + 2 - y^2 - z^2, \end{aligned}$$

and $S_k(\gamma)$ are the Chebyshev polynomials defined by $S_0(\gamma) = 1$, $S_1(\gamma) = \gamma$ and $S_{k+1}(\gamma) = \gamma S_k(\gamma) - S_{k-1}(\gamma)$ for all integer k.

(ii) The universal character ring of the $(-2, 2m+1, 2n)$-pretzel link is reduced for all integers m and n.

The rest of the paper is devoted to the proof of Theorems 1 and 2.

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1. Proof of Theorem 1

Proposition 1.1. *Let $G := \langle a, w \mid u = v \rangle$, where u and v are two words in F_2. Then the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the five polynomials $P_u - P_v$, $P_{ua} - P_{va}$, $P_{uw} - P_{vw}$, $P_{uaw} - P_{vaw}$ and $P_{uwa} - P_{vwa}$.*

Proof. The proof is similar to that of [CS, Prop 1.4.1]. Let I be the ideal in $\mathbb{C}[x, y, z]$ generated by the five polynomials $P_u - P_v$, $P_{ua} - P_{va}$, $P_{uw} - P_{vw}$, $P_{uaw} - P_{vaw}$ and $P_{uwa} - P_{vwa}$. We need to show that $P_{ug} - P_{vg} \in I$ for every $g \in G$. The proof will be based on the identity

$$P_{BAC} + P_{BA^{-1}C} = P_A P_{BC} \tag{1.1}$$

for all matrices A, B, C in $SL_2(\mathbb{C})$, which follows from the identity $A + A^{-1} = P_A I_{2\times 2}$ where $I_{2\times 2}$ is the 2×2 identity matrix.

Let $g_1 := a$ and $g_2 := w$. We first show that $P_{ug} - P_{vg} \in I$ whenever $g = g_{i_1}^{m_1} g_{i_2}^{m_2}$, where i_1, i_2 are distinct positive integers ≤ 2 and $m_1, m_2 \in \mathbb{Z}$. We use induction on the integer $\eta = k_1 + k_2$ where k_j is defined to be $-m_j$ if $m_j \leq 0$ and $m_j - 1$ if $m_j > 0$. If $\eta = 0$ then all the m_j are 0 or 1, so g is equal to $1, a, w, aw$ or wa and hence $P_{ug} - P_{vg} \in I$ by

definition. If $\eta > 0$ then $k_1 > 0$ or $k_2 > 0$. If $k_1 > 0$ then $m_1 \neq 0, 1$. If $m_1 < 0$ then by applying the identity (1.1) we have

$$\begin{aligned} P_{ug} - P_{vg} &= (P_{g_{i_1}} P_{ug_{i_1}g} - P_{ug_{i_1}^2 g}) - (P_{g_{i_1}} P_{vg_{i_1}g} - P_{vg_{i_1}^2 g}) \\ &= P_{g_{i_1}} (P_{ug_{i_1}g} - P_{vg_{i_1}g}) - (P_{ug_{i_1}^2 g} - P_{vg_{i_1}^2 g}) \end{aligned}$$

where $P_{ug_{i_1}g} - P_{vg_{i_1}g}$ and $P_{ug_{i_1}^2 g} - P_{vg_{i_1}^2 g}$ are in I by the induction hypothesis, hence $P_{ug} - P_{vg} \in I$. A similar reduction works if $m_1 > 1$. The case $k_2 > 0$ is similar.

Now let $g \in G$ be arbitrary. We may write g in the form $g_{i_1}^{m_1} \cdots g_{i_r}^{m_r}$ where $i_1, \cdots, i_r$ are integers that are not necessarily distinct. We will prove by induction on r that $P_{ug} - P_{vg} \in I$.

By the case already proved we may assume that $i_1, \cdots, i_r$ are not all distinct. Suppose that $i_k = i_l$ for some $k < l$. Let

$$b = g_{i_1}^{m_1} \cdots g_{i_k}^{m_k}, \quad c = g_{i_{k+1}}^{m_{k+1}} \cdots g_{i_l}^{m_l}, \quad d = g_{i_{l+1}}^{m_{l+1}} \cdots g_{i_r}^{m_r}.$$

Then $g = bcd$. By applying the identity (1.1) we have

$$\begin{aligned} P_{ubcd} - P_{vbcd} &= (P_{ubd} P_c - P_{ubc^{-1}d}) - (P_{vbd} P_c - P_{vbc^{-1}d}) \\ &= (P_{ubd} - P_{vbd}) P_c - (P_{ubc^{-1}d} - P_{vbc^{-1}d}) \end{aligned}$$

But $P_{ubd} - P_{vbd}$ and $P_{ubc^{-1}d} - P_{vbc^{-1}d}$ are in I by the induction hypothesis, and hence $P_{ubcd} - P_{vbcd}$ is also in I. $\square$

Proposition 1.2. *For every words u, v in F_2 one has $P_{uv} = P_{\overleftarrow{u}\overleftarrow{v}}$.*

Proof. It is easy to see from the definition of the operator $\overleftarrow{\cdot}$ that $\overleftarrow{uv} = \overleftarrow{v}\,\overleftarrow{u}$. By [Le1, Lem 3.2.2], for every word s in F_2 we have $P_s = P_{\overleftarrow{s}}$. Hence $P_{uv} = P_{\overleftarrow{uv}} = P_{\overleftarrow{v}\overleftarrow{u}}$. The proposition follows since $P_{\overleftarrow{v}\overleftarrow{u}} = P_{\overleftarrow{u}\overleftarrow{v}}$. $\square$

1.1. **Proof of Theorem 1.** From Proposition 1.1 it follows that the universal character ring of the group $G = \langle a, w \mid r = \overleftarrow{r} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the five polynomials $P_r - P_{\overleftarrow{r}}$, $P_{ra} - P_{\overleftarrow{r}a}$, $P_{rw} - P_{\overleftarrow{r}w}$, $P_{raw} - P_{\overleftarrow{r}aw}$ and $P_{rwa} - P_{\overleftarrow{r}wa}$. By Proposition 1.2 we have

$$\begin{aligned} P_r - P_{\overleftarrow{r}} &= 0, \\ P_{ra} - P_{\overleftarrow{r}a} &= 0, \\ P_{rw} - P_{\overleftarrow{r}w} &= 0, \\ P_{raw} - P_{\overleftarrow{r}aw} &= P_{\overleftarrow{r}wa} - P_{rwa}. \end{aligned}$$

Hence the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the principal ideal generated by the polynomial $P_{raw} - P_{\overleftarrow{r}aw}$.

2. Proof of Theorem 2

2.1. **Proof of part (i).** The fundamental group of the $(-2, 2m+1, 2n)$-pretzel link is

$$\pi := \langle a, b, c \mid bab^{-1} = (ac)^{-m} c (ac)^m, \ a^{-1}ba = (cb)^n b (cb)^{-n} \rangle$$

where a, b, c are meridians depicted in Figure 1.

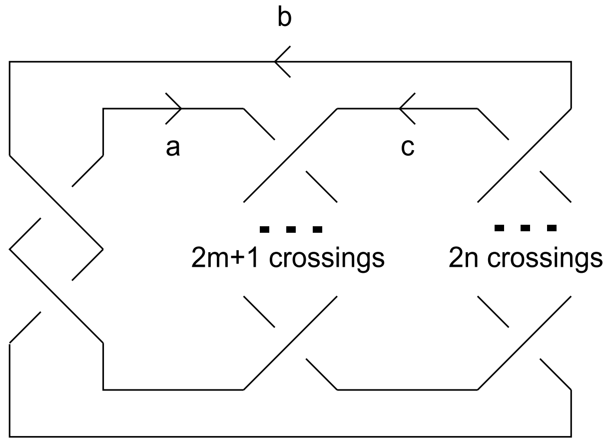

FIGURE 1. The $(-2, 2m+1, 2n)$-pretzel link

The first relation in the group π is $(ac)^m ba = c(ac)^m b$, i.e. $a(ca)^{m-1}cba = ca(ca)^{m-1}cb$. Let $w = (ca)^{m-1}cb$ then $awa = caw$. It implies that $ca = awaw^{-1}$ and $cb = (ca)^{1-m}w = (awaw^{-1})^{1-m}w$. Let $u = (awaw^{-1})^{1-m}w$. Then $cb = u$ and so

$$b = c^{-1}u = awa^{-1}w^{-1}a^{-1}(awaw^{-1})^{1-m}w = a(awaw^{-1})^{-m}w.$$

The second relation in the group π becomes $(awaw^{-1})^{-m}wa = u^n a(awaw^{-1})^{-m}wu^{-n}$, which is equivalent to $u^{n-1}awaw^{-1}a^{-1} = a^{-1}w^{-1}awau^{n-1}$. Therefore

$$\pi = \langle a, w \mid u^{n-1}awaw^{-1}a^{-1} = a^{-1}w^{-1}awau^{n-1}\rangle.$$

Lemma 2.1. *One has $u = \overleftarrow{u}$, i.e. u is palindrome.*

Proof. We first claim that $\overleftarrow{s^k} = \overleftarrow{s}^k$ for all integers k. Indeed, since $\overleftarrow{s}\,\overleftarrow{s^{-1}} = \overleftarrow{s^{-1}s} = 1$ we obtain $\overleftarrow{s^{-1}} = \overleftarrow{s}^{-1}$. If $k \geq 0$ then it is easy to prove by induction on k that $\overleftarrow{s^k} = \overleftarrow{s}^k$. If $k < 0$ then $\overleftarrow{s^k} = \overleftarrow{(s^{-1})^{-k}} = \left(\overleftarrow{s^{-1}}\right)^{-k} = (\overleftarrow{s}^{-1})^{-k} = \overleftarrow{s}^k$.

Applying the identity in the above claim with $s = awaw^{-1}$ and $k = 1-m$ we get

$$\overleftarrow{u} = \overleftarrow{(awaw^{-1})^{1-m}w} = w(w^{-1}awa)^{1-m} = w[w^{-1}(awaw^{-1})^{-m}awa] = (awaw^{-1})^{-m}awa.$$

It implies that $\overleftarrow{u} = (awaw^{-1})^{1-m}w = u$. $\square$

Let $r := u^{n-1}awaw^{-1}a^{-1}$. Then, by Lemma 2.1, we have $\overleftarrow{r} = a^{-1}w^{-1}awa\overleftarrow{u}^{n-1} = a^{-1}w^{-1}awau^{n-1}$. Hence $\pi = \langle a, w \mid r = \overleftarrow{r}\rangle$ and so, by Theorem 1, the universal character ring of π is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the principal ideal generated by the polynomial $P_{raw} - P_{\overleftarrow{r}aw}$, where $x = P_a$, $y = P_w$ and $z = P_{aw}$.

Lemma 2.2. *Suppose the sequence $\{f_k\}_{k=-\infty}^{\infty}$ satisfies the recurrence relation $f_{k+1} = \gamma f_k - f_{k-1}$. Then $f_k = S_{k-1}(\gamma)f_1 - S_{k-2}(\gamma)f_0$, where $S_k(\gamma)$ are the Chebyshev polynomials defined by $S_0(\gamma) = 1$, $S_1(\gamma) = \gamma$ and $S_{k+1}(\gamma) = \gamma S_k(\gamma) - S_{k-1}(\gamma)$ for all integers k.*

Proof. Let $\{g_k\}_{k=-\infty}^{\infty}$ be the sequence defined by $g_k = S_{k-1}(\gamma)f_1 - S_{k-2}(\gamma)f_0$. Then it is easy to see that $g_{k+1} = \gamma g_k - g_{k-1}$. Moreover, since $S_0(\gamma) = 1$ and $S_{-1}(\gamma) = 0$ we have $g_0 = f_0$, $g_1 = f_1$. Therefore $g_k = f_k$. $\square$

Let $\alpha =: P_u$ and $\beta := P_{awaw^{-1}}$.

Proposition 2.3. *One has*

$$\begin{aligned} \alpha &= yS_{m-1}(\beta) - (xz - y)S_{m-2}(\beta), \\ \beta &= xyz + 2 - y^2 - z^2. \end{aligned}$$

Proof. By applying the identity (1.1) and Lemma 2.2 we have

$$\begin{aligned} \beta = P_{awaw^{-1}} &= P_{awa}P_w - P_{awaw} \\ &= (P_{aw}P_a - P_{awa^{-1}})P_w - (P_{aw}P_{aw} - P_{I_2}) \\ &= (zx - y)y - (z^2 - 2), \\ \alpha = P_u &= P_{(awaw^{-1})^{-m}awa} \\ &= P_{(awa)^{-1}(awaw^{-1})^m} \\ &= P_{w^{-1}}S_{m-1}(\beta) - P_{(awa)^{-1}}S_{m-2}(\beta) \\ &= yS_{m-1}(\beta) - (xz - y)S_{m-2}(\beta). \end{aligned}$$

This proves the proposition. □

Proposition 2.4. *One has*

$$P_{raw} - P_{\overleftarrow{r}aw} = (xyz + 4 - x^2 - y^2 - z^2)[(xz - y)S_{n-1}(\alpha) - (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha)].$$

Proof. By applying the identity (1.1) and Lemma 2.2 we have

$$\begin{aligned} P_{raw} &= P_{u^{n-1}awa} = P_{u^n w^{-1}(awaw^{-1})^m w} \\ &= P_{awa}S_{n-1}(\alpha) - P_{w^{-1}(awaw^{-1})^m w}S_{n-2}(\alpha) \\ &= (xz - y)S_{n-1}(\alpha) - (\beta S_{m-1}(\beta) - 2S_{m-2}(\beta))S_{n-2}(\alpha), \\ P_{\overleftarrow{r}aw} &= P_{a^{-1}w^{-1}awau^{n-1}aw} = P_{a^{-1}w^{-1}(awaw^{-1})^m u^n aw} \\ &= P_{a^{-1}w^{-1}awaaw}S_{n-1}(\alpha) - P_{a^{-1}w^{-1}(awaw^{-1})^m aw}S_{n-2}(\alpha) \\ &= P_{a^{-1}w^{-1}awaaw}S_{n-1}(\alpha) - (P_{a^{-1}w^{-1}awaw^{-1}aw}S_{m-1}(\beta) - P_{a^{-1}w^{-1}aw}S_{m-2}(\beta))S_{n-2}(\alpha) \end{aligned}$$

where

$$\begin{aligned} P_{a^{-1}w^{-1}awaaw} &= P_{awa}P_{a^{-1}w^{-1}aw} - P_{a^{-1}w^{-1}(awa)^{-1}aw} \\ &= P_{awa}(P_a P_{w^{-1}aw} - P_{aw^{-1}aw}) - P_{a^{-1}w^{-1}a^{-1}} \\ &= (xz - y)(x^2 - \beta - 1), \\ P_{a^{-1}w^{-1}awaw^{-1}aw} &= P_{awaw^{-1}}P_{a^{-1}w^{-1}aw} - P_{a^{-1}w^{-1}(awaw^{-1})^{-1}aw} \\ &= P_{awaw^{-1}}(P_a P_{w^{-1}aw} - P_{aw^{-1}aw}) - P_{a^{-2}} \\ &= \beta(x^2 - \beta) - (x^2 - 2). \end{aligned}$$

Hence

$$\begin{aligned} P_{\overleftarrow{r}aw} &= (xz - y)(x^2 - \beta - 1)S_{n-1}(\alpha) \\ &\quad - \big((\beta(x^2 - \beta) - (x^2 - 2))S_{m-1}(\beta) - (x^2 - \beta)S_{m-2}(\beta)\big)\, S_{n-2}(\alpha), \end{aligned}$$

and so

$$\begin{aligned} P_{raw} - P_{\overleftarrow{r}aw} &= (\beta + 2 - x^2)[(xz - y)S_{n-1}(\alpha) - ((\beta - 1)S_{m-1}(\beta) - S_{m-2}(\beta))S_{n-2}(\alpha)] \\ &= (\beta + 2 - x^2)[(xz - y)S_{n-1}(\alpha) - (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha)]. \end{aligned}$$

This proves the proposition since $\beta + 2 - x^2 = xyz + 4 - x^2 - y^2 - z^2$. □

Part (i) of Theorem 2 follows from Propositions 2.3 and 2.4.

2.2. **Proof of part (ii).** Recall from Proposition 2.3 that $\alpha = yS_{m-1}(\beta) - (xz-y)S_{m-2}(\beta)$ and $\beta = xyz + 2 - y^2 - z^2$. Let

$$Q(x, y, z) = (xz - y)S_{n-1}(\alpha) - (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha).$$

Then, by Proposition 2.4, $P_{raw} - P_{\overleftarrow{r}aw} = (xyz + 4 - x^2 - y^2 - z^2)Q(x, y, z)$.

Proposition 2.5. *One has*

$$Q(x, y, 0) = (-1)^{(m-1)(n-1)}\, S_{2mn-2m-n-2}(y).$$

Proof. Fix $z = 0$. Then we have $\beta = 2 - y^2,\ \alpha = y(S_{m-1}(\beta) + S_{m-2}(\beta))$ and

$$Q = -[yS_{n-1}(\alpha) + (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha)].$$

Let $y = a + a^{-1}$. Then $\beta = -a^2 - a^{-2}$ and so

$$\begin{aligned}
\alpha &= y(S_{m-1}(\beta) + S_{m-2}(\beta)) \\
&= (a + a^{-1}) \left(\frac{(-a^2)^m - (-a^{-2})^m}{(-a^2) - (-a^{-2})} + \frac{(-a^2)^{m-1} - (-a^{-2})^{m-1}}{(-a^2) - (-a^{-2})} \right) \\
&= (-1)^{m-1}(a^{2m-1} + a^{1-2m}).
\end{aligned}$$

Hence

$$\begin{aligned}
-Q &= yS_{n-1}(\alpha) + (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha) \\
&= (a + a^{-1}) \frac{((-1)^{m-1}a^{2m-1})^n - ((-1)^{m-1}a^{(1-2m)})^n}{(-1)^{m-1}a^{2m-1} - (-1)^{m-1}a^{1-2m}} \\
&\quad + \left(\frac{(-a^2)^{m+1} - (-a^{-2})^{m+1}}{(-a^2) - (-a^{-2})} - \frac{(-a^2)^m - (-a^{-2})^m}{(-a^2) - (-a^{-2})} \right) \\
&\quad \times \frac{((-1)^{m-1}a^{2m-1})^{n-1} - ((-1)^{m-1}a^{1-2m})^{n-1}}{(-1)^{m-1}a^{2m-1} - (-1)^{m-1}a^{1-2m}} \\
&= (-1)^{(m-1)(n-1)}(a + a^{-1}) \frac{a^{(2m-1)n} - a^{(1-2m)n}}{a^{2m-1} - a^{1-2m}} \\
&\quad + (-1)^{m+(m-1)(n-2)} \frac{a^{2m+1} - a^{-(2m+1)}}{a - a^{-1}} \times \frac{a^{(2m-1)(n-1)} - a^{(1-2m)(n-1)}}{a^{2m-1} - a^{1-2m}} \\
&= (-1)^{(m-1)(n-1)} \frac{a^{-2mn+2m+n+1} - a^{2mn-2m-n-1}}{a - a^{-1}} \\
&= (-1)^{(m-1)(n-1)+1}\, S_{2mn-2m-n-2}(y).
\end{aligned}$$

The proposition follows. □

For two polynomials f, g in $\mathbb{C}[x, y, z]$, we say that they are *y-equal*, and write

$$f =_y g$$

if their y-degrees are equal and the coefficients of their highest powers in y are also equal.

Proposition 2.6. *One has*

$$Q(x,y,z) =_y \begin{cases} (-1)^{(m-1)(n-1)} z^2 y^{2mn-2m-n} & \text{if} \quad m \ge 2 \text{ and } n \ge 2, \\ -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n} & \text{if} \quad m \ge 2 \text{ and } n \le 1, \\ z^2 y^{n-2} & \text{if} \quad m = 1 \text{ and } n \ge 3, \\ z^2 - 1 & \text{if} \quad m = 1 \text{ and } n = 2, \\ -y^{2-n} & \text{if} \quad m = 1 \text{ and } n \le 1, \\ (-1)^n y^n & \text{if} \quad m = 0 \text{ and } n \ge 0, \\ 0 & \text{if} \quad m = 0 \text{ and } n = -1, \\ (-1)^{n-1} y^{-(n+2)} & \text{if} \quad m = 0 \text{ and } n \le -2, \\ -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n} & \text{if} \quad m \le -1 \text{ and } n \ge 1, \\ (-1)^{(m-1)(n-1)} y^{2mn-2m-n-2} & \text{if} \quad m \le -1 \text{ and } n \le 0. \end{cases}$$

Proof. We first prove the following result

Lemma 2.7. *One has*
(i) $\alpha =_y (-1)^{m-1} y^{|2m-1|}$.
(ii) $S_m(\beta) - S_{m-1}(\beta) =_y \begin{cases} (-1)^m y^{2m} & \text{if} \quad m \ge 0, \\ (-1)^{m-1} y^{-2(m+1)} & \text{if} \quad m \le -1. \end{cases}$

Proof. (i) Note that $\beta =_y -y^2$. If $m \ge 2$ then

$$S_{m-1}(\beta) =_y S_{m-1}(-y^2) =_y (-y^2)^{m-1} = (-1)^{m-1} y^{2m-2}.$$

Similarly $S_{m-2}(\beta) =_y (-1)^{m-2} y^{2m-4}$. Hence

$$\alpha = yS_{m-1}(\beta) - (xz-y)S_{m-2}(\beta) =_y (-1)^{m-1} y^{2m-1}.$$

If $m = 1$ then $\alpha = y$. If $m = 0$ then $\alpha = xz - y$. If $m \le -1$ then let $m' = -(m+1) \ge 0$. Note that $S_k(\gamma) = -S_{-k-2}(\gamma)$ for all integers k. Hence

$$S_{m-1}(\beta) = -S_{-m-1}(\beta) = -S_{m'}(\beta) =_y -S_{m'}(-y^2) = -(-1)^{m'} y^{2m'} = (-1)^m y^{-2(m+1)}.$$

Similarly $S_{m-2}(\beta) = -S_{m'+1}(\beta) =_y (-1)^{m-1} y^{-2m}$. Hence

$$\alpha = yS_{m-1}(\beta) - (xz-y)S_{m-2}(\beta) =_y (-1)^{m-1} y^{1-2m}.$$

(ii) Similar to (i). □

2.2.1. *The case $m = 0$.* Then $\alpha = xz - y$ and so

$$\begin{aligned} Q &= (xz-y)S_{n-1}(xz-y) - S_{n-2}(xz-y) \\ &= S_n(xz-y) =_y \begin{cases} (-1)^n y^n & \text{if} \quad n \ge 0, \\ 0 & \text{if} \quad n = -1, \\ (-1)^{n-1} y^{-(n+2)} & \text{if} \quad n \le -2. \end{cases} \end{aligned}$$

2.2.2. *The case $m \le -1$.* Then, by Lemma 2.7, $\alpha =_y (-1)^{m-1} y^{1-2m}$ and $S_m(\beta) - S_{m-1}(\beta) =_y (-1)^{m-1} y^{-2(m+1)}$. If $n \ge 2$ then

$$\begin{aligned} S_{n-1}(\alpha) &=_y ((-1)^{m-1} y^{1-2m})^{n-1} = (-1)^{(m-1)(n-1)} y^{-2mn+2m+n-1}, \\ S_{n-2}(\alpha) &=_y ((-1)^{m-1} y^{1-2m})^{n-2} = (-1)^{(m-1)(n-2)} y^{-2mn+4m+n-2}. \end{aligned}$$

It follows that

$$\begin{aligned} (xz-y)S_{n-1}(\alpha) &=_y -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n}, \\ (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha) &=_y (-1)^{m-1} y^{-2(m+1)} (-1)^{(m-1)(n-2)} y^{-2mn+4m+n-2} \\ &= (-1)^{(m-1)(n-1)} y^{-2mn+2m+n-4}. \end{aligned}$$

Hence $Q = (xz-y)S_{n-1}(\alpha) - (S_m(\beta) - S_{m-1}(\beta))S_{n-2}(\alpha) =_y -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n}$.

If $n = 1$ then $Q = xz - y$. If $n = 0$ then $Q = S_m(\beta) - S_{m-1}(\beta) =_y (-1)^{m-1} y^{-2(m+1)}$. Similarly, if $n \le -1$ then $Q =_y (-1)^{(m-1)(n-1)} y^{2mn-2m-n-2}$. Hence

$$Q =_y \begin{cases} -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n} & \text{if} \quad m \le -1 \text{ and } n \ge 1, \\ (-1)^{(m-1)(n-1)} y^{2mn-2m-n-2} & \text{if} \quad m \le -1 \text{ and } n \le 0. \end{cases}$$

If $m \ge 1$ then we write

$$\begin{aligned} Q &= (xz-y)S_{n-1}(\alpha) - ((\beta - 1)S_{m-1}(\beta) - S_{m-2}(\beta))S_{n-2}(\alpha) \\ &= (xz-y)S_{n-1}(\alpha) - ((xyz + 1 - y^2 - z^2)S_{m-1}(\beta) - S_{m-2}(\beta))S_{n-2}(\alpha) \\ &= (xz-y)(S_{n-1}(\alpha) - yS_{m-1}(\beta)S_{n-2}(\alpha)) + ((z^2-1)S_{m-1}(\beta) + S_{m-2}(\beta))S_{n-2}(\alpha) \\ &= (xz-y)(S_{n-1}(\alpha) - (\alpha + (xz-y)S_{m-2}(\beta))S_{n-2}(\alpha)) \\ &\quad + ((z^2-1)S_{m-1}(\beta) + S_{m-2}(\beta))S_{n-2}(\alpha) \\ &= -(xz-y)(S_{n-3}(\alpha) + (xz-y)S_{m-2}(\beta)S_{n-2}(\alpha)) \\ &\quad + ((z^2-1)S_{m-1}(\beta) + S_{m-2}(\beta))S_{n-2}(\alpha) \\ &= [(z^2-1)S_{m-1}(\beta) - ((xz-y)^2 - 1)S_{m-2}(\beta)]S_{n-2}(\alpha) - (xz-y)S_{n-3}(\alpha) \\ &= [(z^2-1)S_{m-1}(\beta) - (-xyz + x^2z^2 - z^2 + 1 - \beta)S_{m-2}(\beta)]S_{n-2}(\alpha) - (xz-y)S_{n-3}(\alpha) \\ &= [z^2 S_{m-1}(\beta) + (xyz - x^2z^2 + z^2 - 1)S_{m-2}(\beta) + S_{m-3}(\beta)]S_{n-2}(\alpha) - (xz-y)S_{n-3}(\alpha) \end{aligned}$$

Let $\delta = z^2 S_{m-1}(\beta) + (xyz - x^2z^2 + z^2 - 1)S_{m-2}(\beta) + S_{m-3}(\beta)$. Then

$$Q = \delta S_{n-2}(\alpha) - (xz-y)S_{n-3}(\alpha).$$

Lemma 2.8. *One has*

$$\delta =_y \begin{cases} (-1)^{m-1} z^2 y^{2m-2} & \text{if} \quad m \ge 2, \\ z^2 - 1 & \text{if} \quad m = 1, \\ (-1)^m y^{2-2m} & \text{if} \quad m \le 0. \end{cases}$$

2.2.3. *The case $m = 1$.* In this case $\alpha = y$ and so

$$Q = (z^2-1)S_{n-2}(y) - (xz-y)S_{n-3}(y) = z^2 S_{n-2}(y) - xzS_{n-3}(y) + S_{n-4}(y).$$

Hence

$$Q =_y \begin{cases} z^2 y^{n-2} & \text{if} \quad n \ge 3, \\ z^2 - 1 & \text{if} \quad n = 2, \\ -y^{2-n} & \text{if} \quad n \le 1. \end{cases}$$

2.2.4. *The case* $m \geq 2$. Then, by Lemmas 2.7 and 2.8, $\alpha =_y (-1)^{m-1} y^{2m-1}$ and $\delta =_y (-1)^{m-1} z^2 y^{2m-2}$. By similar arguments as in the case $m \leq -1$, we obtain

$$Q =_y \begin{cases} (-1)^{(m-1)(n-1)} z^2 y^{2mn-2m-n} & \text{if} \quad m \geq 2 \text{ and } n \geq 2, \\ -(-1)^{(m-1)(n-1)} y^{-2mn+2m+n} & \text{if} \quad m \geq 2 \text{ and } n \leq 1. \end{cases}$$

This completes the proof of Proposition 2.6. □

From Propositions 2.5 and 2.6, we have

(i) If $Q(x, y, z)$ has non-trivial repeated factors then so is $Q(0, y, z)$. Moreover, if $R(y, z)$ is a non-trivial repeated factor of $Q(0, y, z)$ then the coefficient of the highest power of y in $R(y, z)$ is a divisor of z.

(ii) The difference of the y-degrees of $Q(0, y, z)$ and $Q(0, y, 0) = \pm S_{2mn-2m-n-2}(y)$ is at most 2.

Let us now prove part (ii) of Theorem 2. The goal is to show that

$$P_{raw} - P_{\overleftarrow{r}aw} = (xyz + 4 - x^2 - y^2 - z^2) Q(x, y, z)$$

does not have any non-trivial repeated factors.

Suppose that $Q(x, y, z)$ has non-trivial repeated factors. Then $Q(0, y, z)$ also has non-trivial repeated factors. Let $R(y, z)$ be a non-trivial repeated factor of $Q(0, y, z)$. Note that the coefficient of the highest power of y in $R(y, z)$ is a divisor of z. If R has y-degree 0 then $R = \pm z$. It implies that z is a divisor of $Q(0, y, z)$ and so $Q(0, y, 0) = \pm S_{2mn-2m-n-2}(y) = 0$. Hence $2mn - 2m - n - 2 = -1$, i.e. ($m = 0$ and $n = -1$) or ($m = 1$ and $n = 3$). If $m = 0$ and $n = -1$ then $\alpha = xz - y$ and so $Q(x, y, z) = 0$. If $m = 1$ and $n = 3$ then $\alpha = y$ and so $Q(x, y, z) = z(zy - x)$ does not have any non-trivial repeated factors.

We consider the case that R has y-degree $k \geq 1$. Let r_k be the coefficient of y^k in $R(y, z)$. If $r_k = \pm 1$ then $R(y, 0)$ is a non-trivial repeated factor of $Q(0, y, 0) = \pm S_{2mn-2m-n-2}(y)$. This is impossible since $S_{2mn-2m-n-2}(y)$ does not have any non-trivial repeated factors. Hence $r_k = \varepsilon z$, where $\varepsilon = \pm 1$, and so $R(y, z) = \varepsilon z y^k + r_{k-1} y^{k-1} + \cdots$. Since the difference of the y-degrees of $Q(0, y, z)$ and $Q(0, y, 0)$ is at most 2, the y-degree of $R(y, 0)$ is exactly $k - 1$. If $k \geq 2$ then $R(y, 0)$ is a non-trivial repeated factor of $Q(0, y, 0) = \pm S_{2mn-2m-n-2}(y)$, which is impossible. Hence $k = 1$ and so $R(y, z) = \varepsilon z y + r_0(z)$ where $r_0(0) \neq 0$. We have

$$Q(0, y, z) = -[y S_{n-1}(\alpha \mid_{x=0}) + (S_m(\beta \mid_{x=0}) - S_{m-1}(\beta \mid_{x=0})) S_{n-2}(\alpha \mid_{x=0})]$$

where $\beta \mid_{x=0} = 2 - y^2 - z^2$ and $\alpha \mid_{x=0} = y[S_{m-1}(\beta \mid_{x=0}) + S_{m-2}(\beta \mid_{x=0})]$. It implies that $Q(0, y, z)$ contains even powers of z only. Since $R(y, z) = \varepsilon z + r_0(z)$ is a non-trivial repeat factor of $Q(0, y, z)$, so is $R(y, -z) = \varepsilon(-z) + r_0(-z)$. If $R(y, -z) \neq -R(y, z)$, then $R(y, z)$ and $R(y, -z)$ are distinct non-trivial repeated factors in the prime factorization of $Q(0, y, z)$ in the UFD $\mathbb{C}[y, z]$. It implies that the difference of the y-degrees of $Q(0, y, z)$ and $Q(0, y, 0)$ is at least 4, a contradiction. Hence $R(y, -z) = -R(y, z)$, which means that $r_0(-z) = -r_0(z)$, i.e. $r_0(z)$ is an odd polynomial in z. This contradicts the condition that $r_0(0) \neq 0$. Therefore $Q(x, y, z)$ does not have any non-trivial repeated factors.

It remains to show that $xyz + 4 - x^2 - y^2 - z^2$ is not a divisor of $Q(x, y, z)$ unless $Q(x, y, z) \equiv 0$. From Proposition 2.6, it is easy to see that $Q(x, y, z) \equiv 0$ if and only if $m = 0$ and $n = -1$. Suppose $Q(x, y, z) \not\equiv 0$. If $m = 1$ and $n = 3$ then $Q(x, y, z) = z(zy - x)$. Otherwise $Q(x, y, 0) = \pm S_{2mn-2m-n-2}(y) \not\equiv 0$ is not divisible by $4 - x^2 - y^2$. It

implies that $Q(x,y,z)$ is not divisible by $xyz+4-x^2-y^2-z^2$. Therefore $P_{raw}-P_{\overleftarrow{raw}} = (xyz+4-x^2-y^2-z^2)Q(x,y,z)$ does not have any non-trivial repeated factors and so the universal character ring of the $(-2,2m+1,2n)$-pretzel link is reduced for all integers m and n.

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